BELYI MAPS AND DESSINS D'ENFANTS
LECTURE 14

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CONTENTS

I. REVIEW

Last time we:

(1) Defined a Fuchsian triangle group as the set of orientation-preserving isometries of $\mathfrak{D}$ generated by the rotations about the vertices.
(2) Noted that, given a hyperbolic triangle with angles $\pi/a, \pi/b, \pi/c$, then the triangle group $\Delta(a, b, c)$ has presentation

$$\langle \delta_a, \delta_b, \delta_c \mid {\delta_a}^a = {\delta_b}^b = {\delta_c}^c = \delta_a\delta_b\delta_c = 1 \rangle .$$

(3) Discovered that the modular group $\Gamma(1) = \mathrm{PSL}_2(\mathbb{Z})$ is a triangle group, namely $\Delta(2, 3, \infty)$. We showed this by constructing a fundamental domain for $\Gamma(1)$, namely the triangle with vertices at $i, e^{2\pi i/6}$, and ∞, together with its reflection across the imaginary axis.

II. THE MODULAR GROUP AS A TRIANGLE GROUP

II.1. Subgroups and congruence subgroups.

Proposition 1. *Let Γ and Γ' be Fuchsian groups. Suppose that $\Gamma' \leq \Gamma$ and $[\Gamma : \Gamma'] = n$. Let $\gamma_1, \ldots, \gamma_n \in \Gamma$ be a set of right coset representatives of $\Gamma' \backslash \Gamma$. Let Q be a hyperbolic polygon that is a fundamental domain for Γ. Then*

$$D := \bigcup_{j=1}^{n} \gamma_j(Q)$$

is a fundamental domain for Γ'.

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Remark 2. In other words, if we know a fundamental domain Q for a Fuchsian group Γ and $\Gamma' \leq \Gamma$, we can obtain a fundamental domain for Γ' by translating Q by a set of coset representatives.

An important class of subgroups of $\Gamma(1)$ are so-called principal congruence subgroups. For $N \in \mathbb{Z}_{\geq 1}$, the principal congruence subgroup $\Gamma(N)$ is the kernel of the reduction map

$$\Gamma(1) = \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})$$

that reduces the entries of a matrix modulo N. In other words, $\Gamma(N)$ fits into a short exact sequence

$$1 \to \Gamma(N) \to \Gamma(1) \to \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z}) \to 1\,.$$

$\Gamma(2)$ turns out to be of particular interest. We will show that $\Gamma(2) \cong \Delta(\infty,\infty,\infty)$, the free group on two generators.

Lemma 3. *Let $q = p^r$ be a prime power. Then*

$$\#\mathrm{GL}_m(\mathbb{F}_q) = (q^m - 1)(q^m - q)\cdots(q^m - q^{m-1})\,.$$

Corollary 4. $\#\mathrm{PSL}_2(\mathbb{Z}/2\mathbb{Z}) = (2^2-1)(2^2-2) = 6$.

Proof. Note that $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ since the only possibilities for the value of the determinant are 0 and 1. Also note that the center of $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ is trivial, so $\mathrm{PSL}_2(\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$. □

In order to apply the above proposition to determine a fundamental domain, we must find a set of coset representatives for $\Gamma(2)\backslash\Gamma(1)$. Recall that the reflections of the $2,3,\infty$ triangle are

$$\tau_3 : z \mapsto -\bar{z}, \quad \tau_2 : z \mapsto -\bar{z}+1, \quad \tau_\infty : z \mapsto 1/\bar{z}\,.$$

(To see this: to get a reflection over the imaginary axis, we first reflect over the real axis, then rotate 180°. To get a rotation across the line $\mathrm{Re}(z) = 1/2$, do the above, then translate by 1. Finally, note that $\bar{z} = 1/z$ for points on the unit circle, so $1/\bar{z}$ fixes the unit circle.) [Show picture on p. 124 of GGD.] We can find the corresponding reflections by composing these, obtaining

$$\begin{aligned}
\delta_2 = \tau_\infty \circ \tau_3 : z \mapsto -1/z \qquad & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
\delta_3 = \tau_\infty \circ \tau_2 : z \mapsto \frac{1}{-z+1} \qquad & \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \\
\delta_\infty = \tau_3 \circ \tau_2 : z \mapsto z - 1 \qquad & \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

are generators for $\Gamma(1)$. One can show that the following is a set of representatives

$$\mathrm{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \delta_3 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad {\delta_3}^2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\delta_\infty = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \delta_\infty\delta_3 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Rather than choosing the usual fundamental domain for $\Gamma(1)$, we will instead take the triangle with vertices $i, e^{2\pi i/6}$ and ∞, toegether with its reflection over the vertical line

$\mathrm{Re}(z) = 1/2$. We denote this fundamental domain by Q'. [Show picture on p. 124 of GGD. again.] By the proposition above, then

$$D = Q' \cup \delta_3(Q') \cup {\delta_3}^2(Q') \cup \delta_\infty(Q') \cup \delta_\infty\delta_3(Q') \cup \delta_2(Q')$$

is a fundamental domain for $\Gamma(2)$. Observe that

$$Q' \cup \delta_3(Q') \cup {\delta_3}^2(Q')$$

is a triangle $\widetilde{T}$ with vertices at $0, 1$, and ∞. Moreover, D is the union of $\widetilde{T}$ with its reflection $\widetilde{T}^-$ across the imaginary axis. Since $\Gamma(2)$ identifies the sides of D in the same way that the triangle group $\Delta(\infty,\infty,\infty)$ does, then $\Gamma(2) \cong \Delta(\infty,\infty,\infty)$.

Proposition 5. $\Gamma(2) \cong \Delta(\infty,\infty,\infty)$

As usual, $\Delta(\infty,\infty,\infty)$ is generated by rotations about the vertices of the triangle $\widetilde{T}$. Using this geometric description we compute that these rotations are given by the matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$$

so by the above, these three matrices generate $\Gamma(2)$.

III. Monodromy and Fuchsian groups

III.1. **Revisiting results in the language of Fuchsian groups.** Many results that we stated in general for covering spaces or morphisms of Riemann surfaces can be reinterpreted in the case of a morphism between quotients of $\mathfrak{H}$ by Fuchsian groups.

In a previous lecture, we proved the following result.

Proposition 6. *Let X_1 and X_2 be Riemann surfaces uniformized by Fuchsian groups Γ_1 and Γ_2 acting freely on $\mathfrak{H}$, so $X_1 \cong \Gamma_1\backslash\mathfrak{H}$ and $X_2 \cong \Gamma_2\backslash\mathfrak{H}$. Then $X_1 \cong X_2$ iff Γ_1 and Γ_2 are conjugate in* $\mathrm{PSL}_2(\mathbb{R})$, *i.e., there exists $T \in \mathrm{PSL}_2(\mathbb{R})$ such that $T\,\Gamma_1 T^{-1} = \Gamma_2$.*

$$\begin{tikzcd} \mathfrak{H} \arrow[r, "T"] \arrow[d, "p_1"] & \mathfrak{H} \arrow[d, "p_2"] \\ \Gamma_1\backslash\mathfrak{H} \arrow[r, "\varphi"'] & \Gamma_2\backslash\mathfrak{H} \end{tikzcd}$$

Proposition 7. *Let Γ be a Fuchsian group acting freely on $\mathfrak{H}$. Then*

$$\mathrm{Aut}(\Gamma\backslash\mathfrak{H}) \cong N(\Gamma)/\Gamma$$

where

$$N(\Gamma) = \{T \in \mathrm{PSL}_2(\mathbb{R}) : T\Gamma T^{-1} = \Gamma\}$$

is the normalizer of Γ in $\mathrm{PSL}_2(\mathbb{R})$. *In particular, when Γ is a normal subgroup of* $\mathrm{PSL}_2(\mathbb{R})$, *then*

$$\mathrm{Aut}(\Gamma\backslash\mathfrak{H}) \cong \mathrm{PSL}_2(\mathbb{R})/\Gamma\,.$$

Proof idea. Taking $\Gamma_1 = \Gamma_2$ in the previous proposition, we obtain a surjection $N(\Gamma) \to \mathrm{Aut}(\Gamma\backslash\mathfrak{H})$ whose kernel is exactly Γ. □

In the material on covering spaces, we discussed the following result.

Theorem 8. *Let G be a finite group acting faithfully on a Riemann surface X. Then $G\backslash X$ can be given the structure of a Riemann surface, and the quotient map $\pi : X \to G\backslash X$ is holomorphic of degree $\#G$ and $e_P(\pi) = \#\mathrm{Stab}_G(P)$ for all $P \in X$.*

Corollary 9. *Let G be a finite group acting faithfully on a compact, connected Rieman surface X, let $Y = G\backslash X$ and let $\pi : X \to Y$ be the quotient map. Suppose that π has k ramification values $y_1, \ldots, y_k \in Y$ such that π has ramification index r_i at each of the $\#G/r_i$ points above y_i. Then*

$$\begin{aligned} 2g(X) - 2 &= \#G(2g(G\backslash X) - 2) + \sum_{i=1}^{k} \frac{\#G}{r_i}(r_i - 1) \\ &= \#G\left(2g(G\backslash X) - 2 + \sum_{i=1}^{k}\left(1 - \frac{1}{r_i}\right)\right). \end{aligned}$$

Remark 10. As mentioned previously, if X is a Riemann surface of genus $g \geq 2$, then $\mathrm{Aut}(X)$ is finite by Hurwitz's theorem on automorphisms. Thus in this case, every group G acting on X must factor through a finite quotient. (In other words, we can basically assume that G is finite.)

Recall that the congruence subgroup $\Gamma(N)$ fits into the short exact sequence

$$1 \to \Gamma(N) \to \Gamma(1) \xrightarrow{\pi} \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z}) \to 1$$

where π is the group homomorphism that reduces the entries of a matrix mod N. We saw that $\Gamma(2)\backslash\mathfrak{H}$ had genus 0, since it was a triangle group. We now use the above corollary to give an expression for $\Gamma(N)\backslash\mathfrak{H}$.

Proposition 11. *Suppose $N \geq 2$ and let g_N be the genus of $\Gamma(N)\backslash\mathfrak{H}$. Then*

$$2g_N - 2 = [\Gamma(1) : \Gamma(N)]\left(-2 + \left(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{N}\right)\right),$$

where $2, 3, N$ are the ramification indices above the three branch values $i, e^{2\pi i/6}$, and ∞.

Proof sketch. By results on universal covering spaces, we have the following commutative diagram.

$$\begin{tikzcd} \mathfrak{H} \arrow[d] \arrow[dr] & \\ \Gamma(N)\backslash\mathfrak{H} \arrow[r, "F"'] & \Gamma(1)\backslash\mathfrak{H} \cong \mathbb{P}^1 \setminus \{\infty\} \end{tikzcd}$$

Moreover, we have

$$\Gamma(1)\backslash\mathfrak{H} \cong \frac{\Gamma(N)\backslash\mathfrak{H}}{\Gamma(N)\backslash\Gamma(1)}$$

so the map F is simply quotienting by the action of

$$\Gamma(N)\backslash\Gamma(1) \cong \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})$$

on $\Gamma(N)\backslash\mathfrak{H}$. Let m_1, m_2, m_3 be the ramification indices above the three branch values $i, e^{2\pi i/6}$, and ∞. Since $N \geq 2$, one can show that $\Gamma(N)$ is torsion-free. This in turn implies that $m_1 = 2$ and $m_2 = 3$. By examining the compactification of $\Gamma(N)\backslash\mathfrak{H}$ at ∞, one can show that $m_3 = N$. (For details, see Example 2.40 of GGD or Diamond and Shurman.) Applying the version of Riemann-Hurwitz for quotient maps, the result follows. $\square$

III.2. **Monodromy via Fuchsian groups.** Recall that, given a morphism $F : X \to Y$, the monodromy of F describes the action of the fundamental group $\pi_1(Y)$ on a fiber of F. We can reinterpret this in terms of Fuchsian groups as well.

Let $F : X \to Y$ be a morphism of Riemann surfaces. Let Y^* be Y without the ramification values of F, and let $X^* = F^{-1}(Y^*)$. Applying the uniformization theorem, we obtain Fuchsian groups $\Gamma_X \leq \Gamma_Y$ such that

$$X^* \cong \Gamma_X \backslash \mathfrak{H} \qquad Y^* \cong \Gamma_Y \backslash \mathfrak{H}$$

as well as a morphism $G : \Gamma_X \backslash \mathfrak{H} \to \Gamma_Y \backslash \mathfrak{H}$. Since $\mathfrak{H} \to \Gamma_Y \backslash \mathfrak{H} \cong Y^*$ is the universal cover of Y^*, then

$$\pi_1(Y^*) \cong \Gamma_Y .$$

Given $y \in Y$, then y corresponds to $[z_0]_{\Gamma_Y} \in \Gamma_Y \backslash \mathfrak{H}$ for some $z_0 \in \mathfrak{H}$. (Here $[\cdot]_{\Gamma_Y}$ denotes the equivalence class modulo the action of Γ_Y.) Moreover, by commutativity of the diagram

$$\begin{tikzcd}
\mathfrak{H} \arrow[d, "\varphi_X"'] \arrow[dr, "\varphi_Y"] & \\
\Gamma_X \backslash \mathfrak{H} \arrow[r, "G"'] & \Gamma_Y \backslash \mathfrak{H}
\end{tikzcd}$$

given $y \in Y$, the fiber $G^{-1}(y)$ is

$$\{[\beta(z_0)]_{\Gamma_X} : \beta \in \Gamma_X \backslash \Gamma_Y\}$$

where β ranges over a set of right coset representatives for $\Gamma_X \backslash \Gamma_Y$. Thus we have a bijection

$$\begin{aligned} \Phi : \Gamma_X \backslash \Gamma_Y &\to G^{-1}(y) \\ \Gamma_X \beta &\mapsto [\beta(z_0)]_{\Gamma_X} . \end{aligned}$$

We want to reinterpret the monodromy representation in terms of the groups Γ_X and Γ_Y using the above bijection. Let

$$\rho : \pi_1(Y) \to \mathrm{Sym}(G^{-1}(y))$$

be the monodromy representation of G. Given $\gamma \in \Gamma_Y$, we describe the permutation $\rho(\gamma)$ as folows.